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LAGRANGIAN-LIKE EQUATIONS OF HYDRODYNAMICS WITH MASS DIFFUSION

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In an inertial reference frame, the Eulerian representation of the inviscid hydrodynamics equations is

$$\frac{\partial \rho}{\partial t} + \frac{\partial G}{\partial x} = 0 \tag{1}$$

$$\frac{\partial \rho \varepsilon}{\partial t} + \frac{\partial G \varepsilon}{\partial x} + p \frac{\partial w}{\partial x} = 0 \tag{2}$$

$$\frac{\partial \rho w}{\partial t} + \frac{\partial G w}{\partial x} + \frac{\partial p}{\partial x} = 0 \tag{3}$$

where ε is the specific internal energy, w is the fluid velocity, p is the pressure, ρ is the density, and G is the mass flux. Equations (1) and (3) are conservation equations for mass and momentum.

An equation for specific kinetic energy is obtained by multiplying Eq. (3) by w and this gives

$$\frac{\partial \rho k}{\partial t} + \frac{\partial G k}{\partial x} + w \frac{\partial p}{\partial x} = 0 \tag{4}$$

where $K = \frac{1}{2} w^2$.

The sum of Eqs. (2) and (4) give a conservation equation for total energy

$$\frac{\partial}{\partial t} \rho(\varepsilon + K) + \frac{\partial}{\partial x} G(\varepsilon + k) + \frac{\partial p w}{\partial x} = 0 \tag{5}$$

The mass flux G is usually given by ρw; here we consider an additional flux given by the diffusion of mass due, for example, to molecular action or turbulence and this G is given by

$$G = \rho w + F \tag{6}$$

where

$$F = - D \frac{\partial \rho}{\partial x} . \tag{7}$$

In the absence of F, the equations are transformed to a Lagrangian reference frame by the operator

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + w \frac{\partial f}{\partial x} . \tag{8}$$

Under this operator, mass is conserved in control volumes that move with the fluid velocity, $w = \frac{dx}{dt}$.

In the presence of F we define a new operator

$$\frac{\delta f}{\delta t} = \frac{\partial f}{\partial t} + S \frac{\partial f}{\partial x} \tag{9}$$

where

$$S = w + \frac{F}{\rho} \tag{10}$$

And note that the two operators are related by

$$\frac{\delta f}{\delta t} = \frac{df}{dt} + \frac{F}{\rho} \frac{\partial f}{\partial x} \tag{11}$$

Using Eq. (9), we transform Eqs. (1)-(3) to the frame of an observer moving with velocity S.

$$\frac{\delta \rho}{\delta t} = - \rho \frac{\partial S}{\partial x} \tag{12}$$

$$\rho \frac{\delta \varepsilon}{\delta t} = - p \frac{\partial w}{\partial x} \tag{13}$$

$$\rho \frac{\delta w}{\delta t} = - \frac{\partial p}{\partial x} \tag{14}$$

We substitute x for f in Eqs. (8) and (9) to obtain

$$\frac{dx}{dt} = w \tag{15}$$

$$\frac{\delta x}{\delta t} = S \tag{16}$$

The total energy equation is

$$\rho \frac{\delta}{\delta t} (\varepsilon + K) = - \frac{\partial pw}{\partial x} \tag{17}$$

We may associate a volume element with Eq. (9) by analogy with the volume element associated with Eq. (8)

$$\frac{1}{V} \frac{\delta V}{\delta t} = \frac{\partial S}{\partial x} \tag{18}$$

and note that Eqs. (12) and (18) combine to give

$$\frac{\delta}{\delta t} \rho V = 0 \tag{19}$$

We now transform to a reference frame that is uniformly accelerating

$$\frac{d^2y}{dt^2} = \frac{d^2x}{dt^2} - g \tag{20}$$

In this new frame, Eqs. (12)-(14) become

$$\frac{\delta \rho}{\delta t} = - \rho \frac{\partial S}{\partial y} \tag{21}$$

$$\rho \frac{\delta \varepsilon}{\delta t} = - p \frac{\partial u}{\partial y} \tag{22}$$

$$\rho \frac{\delta u}{\delta t} = - \frac{\partial p}{\partial y} - \rho g \tag{23}$$

and the first integral of Eq. (20) is

$$u = \frac{dy}{dt} = \frac{dx}{dt} - gt + \dot{y}_o - \dot{x}_o \tag{24}$$

The energy equation becomes

$$\rho \frac{\delta}{\delta t} (\varepsilon + \frac{1}{2} u^2) + ug\rho = - \frac{\partial \rho u}{\partial y}$$

Using Eqs. (24) and (11), this becomes

$$\rho \frac{\delta}{\delta t} (\varepsilon + \frac{1}{2} u^2) + g\rho (\frac{\delta y}{\delta t} - \frac{F}{\rho}) = - \frac{\partial \rho u}{\partial y}$$

which suggests two new energy terms

$$\rho \frac{\delta \phi}{\delta t} = \rho g \frac{\delta y}{\delta t} = \rho g u + gF \tag{25}$$

$$\rho \frac{\delta k}{\delta t} = - gF \tag{26}$$

With ϕ and k so defined, the energy equation becomes

$$\rho \frac{\delta}{\delta t} (\varepsilon + \frac{1}{2} u^2 + \phi + k) = - \frac{\partial \rho u}{\partial y} \tag{27}$$

Equation (26) was identified because it can be associated with a source of turbulent kinetic energy. We could also have ignored this equation by adding Eqs. (25) and (26)

$$\rho \frac{\delta}{\delta t} (\phi + k) = \rho g u \tag{28}$$

and associated $\phi + k$ with "potential" energy.

As it is, we associate the change of potential energy (Eq. (25)) with a mass flux, due to advection and diffusion, in a "gravitational" field. We note that the diffusive mass flux leads to a change in k. The ρgu term states that fluid rising in our accelerated frame gains potential energy. Similarly, fluid diffusing upward ($g > 0$, $F > 0$) gains potential energy and loses k-energy.

Equations (25) and (26) were not known in our original inertial system. To gain further insight it will be useful to transform them back to the inertial system.

But first, let us simplify our transformation. We think of transforming our observer to a coordinate system that is accelerating with the local fluid acceleration, $\dot{w}$. Thus we assume

$$\frac{dw}{dt} = g$$

or

$$\frac{du}{dt} = 0$$

Equations (21) and (22) remain the same while Eq. (23) becomes

$$\rho g = - \frac{\partial p}{\partial y} \ .$$

It's not too surprising to find that the acceleration is provided by the pressure gradient.

To simplify the transformation, we assume that at t = 0 the y and x coordinate systems coincide and have the same velocity.

Then Eq. (24) becomes

$$u = w - \dot{w}t$$

and in the inertial system the potential energy equation is

$$\rho \frac{\delta\Phi}{\delta t} = \rho\dot{w}(w - \dot{w}t) + gF$$

$$= \rho \frac{dK}{dt} - \rho(\dot{w})^2 t + gF \qquad (29)$$

Equation (29) gives the rate of change of potential energy in an inertial system.

The k-equation transforms back to the inertial system unchanged and we have

$$\rho \frac{\delta}{\delta t} (\phi + k) = \rho \dot{w}(w - \dot{w}t)$$

which we intuitively know to be zero since we can't create energy merely by transforming between coordinate system.

We confirm intuition by noting that $w - \dot{w}t = 0$; since $w = 0$ at $t = 0$ and since $\dot{w}$ = constant, $w = \int \dot{w}\, dt = \dot{w}t$.

From the transformation we have found a previously unknown relation between mean kinetic energy and acceleration, in the inertial system:

$$\frac{dK}{dt} = \left(\frac{dw}{dt}\right)^2 t$$

and we restate the newly found energy relation in the inertial system

$$\frac{\delta}{\delta t} (\phi + k) = 0$$

$$\rho \frac{\delta k}{\delta t} = - gF \ .$$

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